Large population limits of Markov processes on random networks

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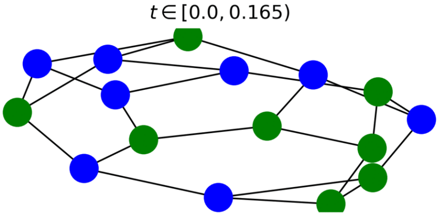
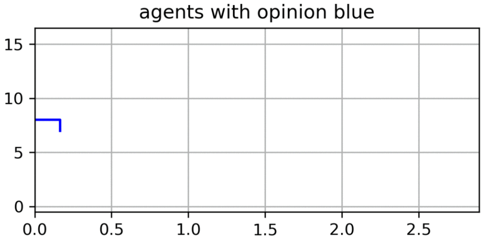
The model

- N **agents**, represented by nodes in a network
- each agent has one of M **discrete states** (e.g., opinion)





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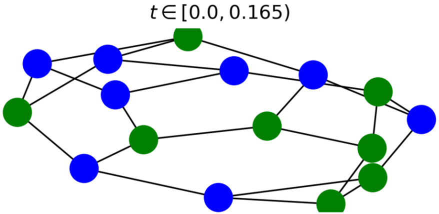

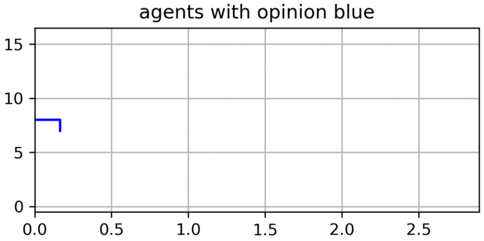

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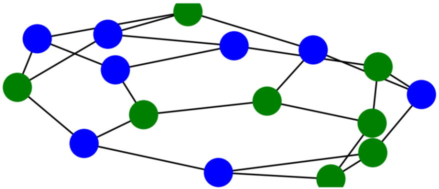
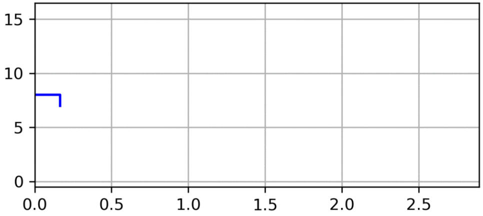
$$r_{m,n} \frac{\#\{\text{neighbors with state } n\}}{\text{node degree}} + \tilde{r}_{m,n}$$

  - model parameters: $r_{m,n}, \tilde{r}_{m,n} \geq 0$.

$t \in [0.0, 0.165)$



agents with opinion blue



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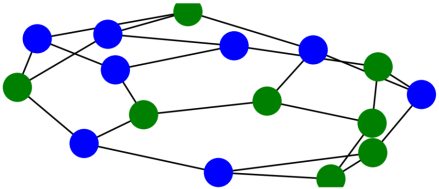
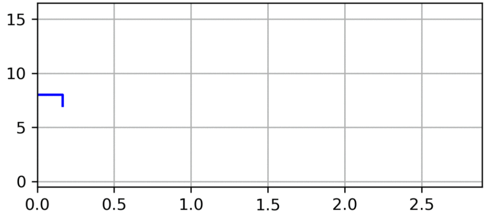
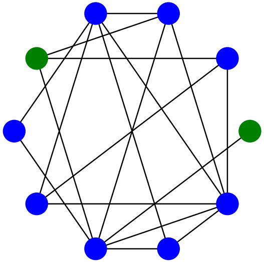
- important observable: **shares** (percentages) **of each state in the system** $c \in [0,1]^M$

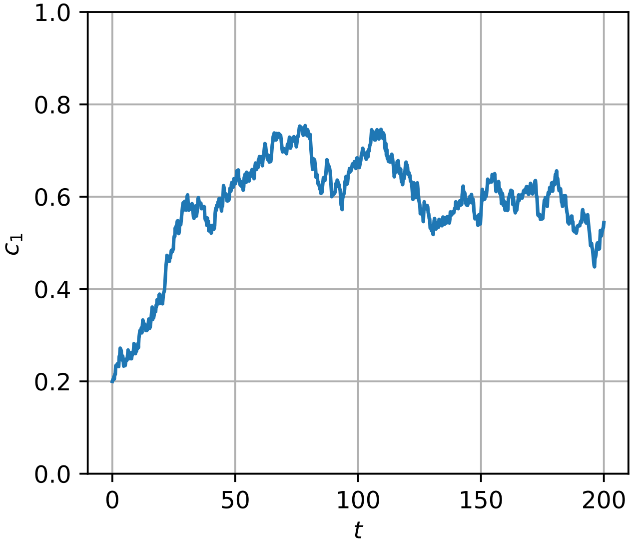
  $\rightarrow$ collective variable?

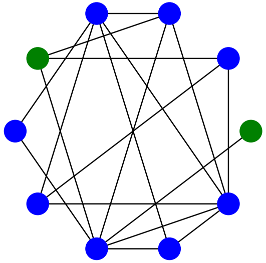






Erdős–Rényi random graph





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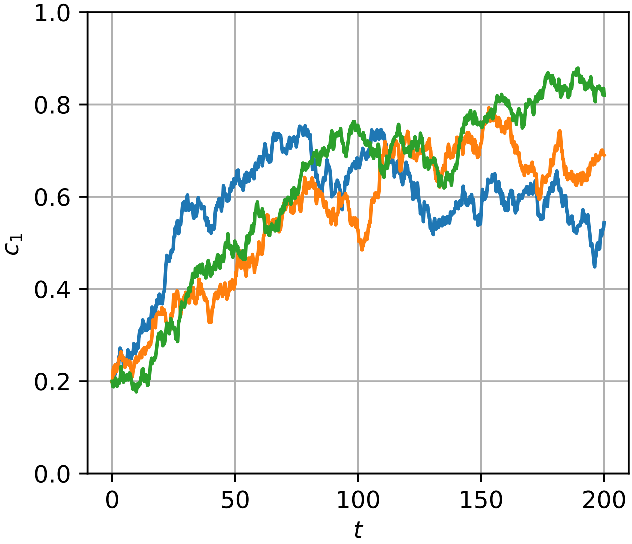

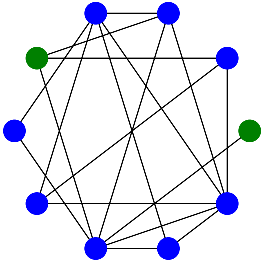

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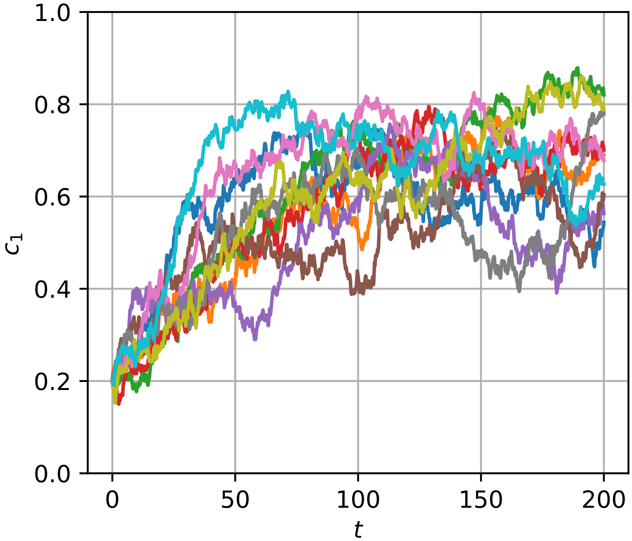

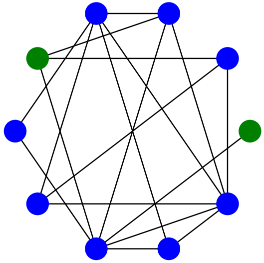

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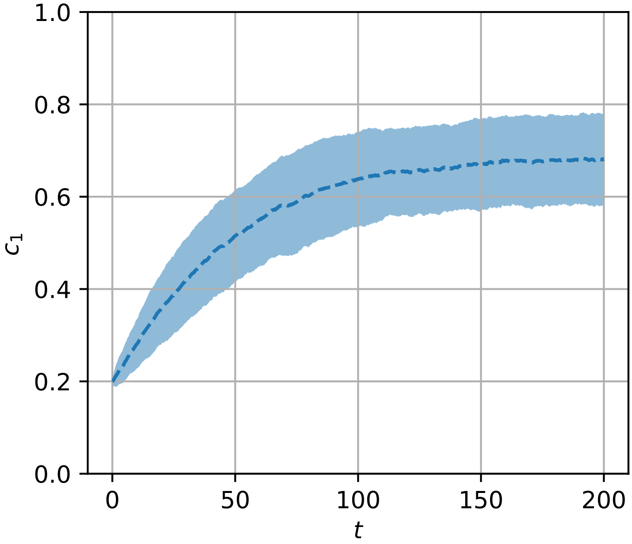

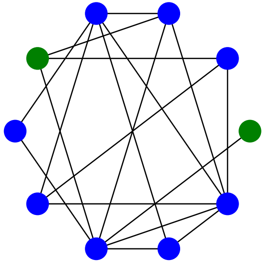

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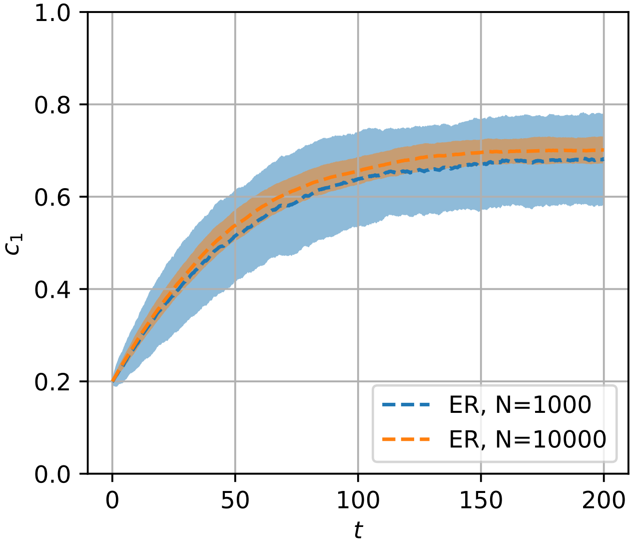

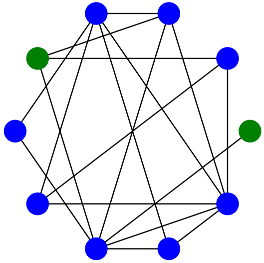

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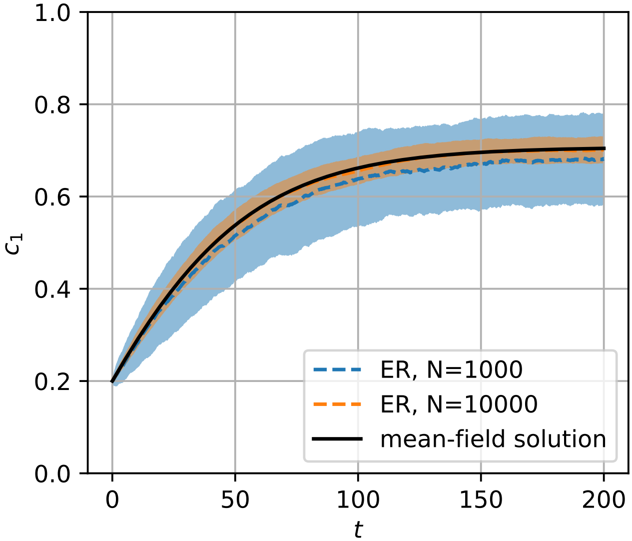
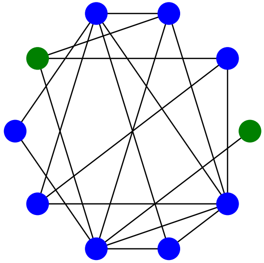


mean-field equation (Kurtz 1978, L et al.)

$$\frac{\mathrm{d}}{\mathrm{d}t}c(t) = \sum_{m \neq n} c_m(t)\Big(r_{m,n}c_n(t) + \tilde{r}_{m,n}\Big)(e_n - e_m)$$



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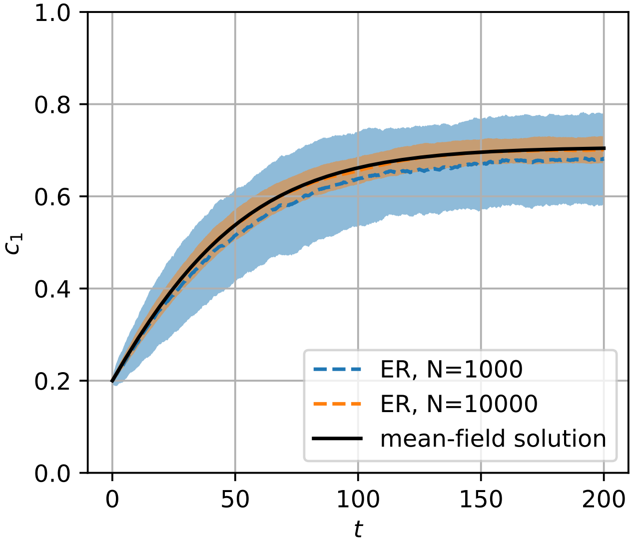

Theorem

If $p \gg \log(N)/N$:

$$\sup_{0\le s\le t} \|C^N(t) - c(t)\| \xrightarrow[N\to\infty]{P} 0.$$

Proof: Law of large numbers, concentration inequalities, Gronwall's lemma

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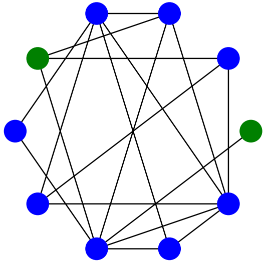

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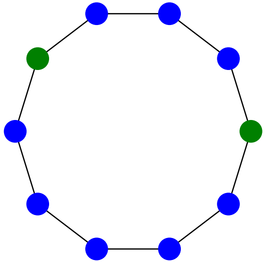
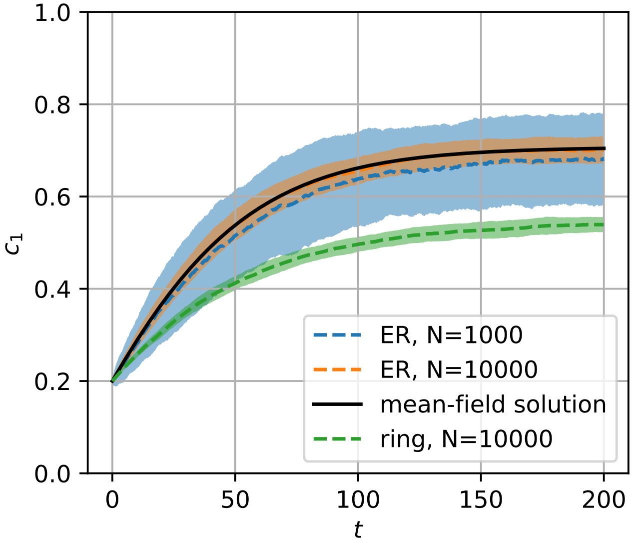


ring graph



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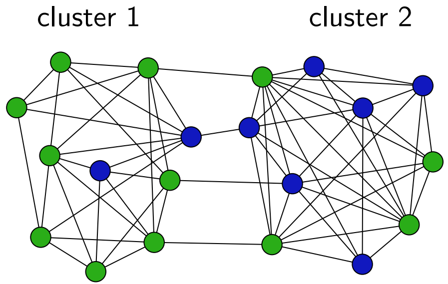
Stochastic block model



$C_{(1,1)} = \frac{2}{20}$

$C_{(2,1)} = \frac{8}{20}$

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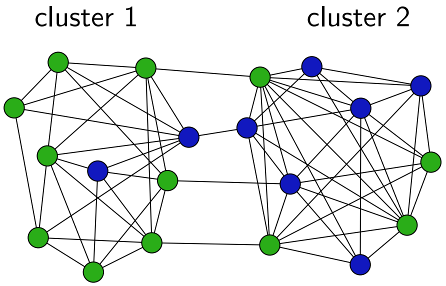

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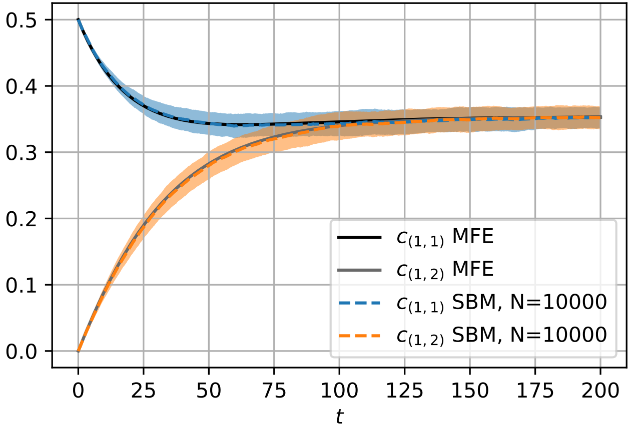

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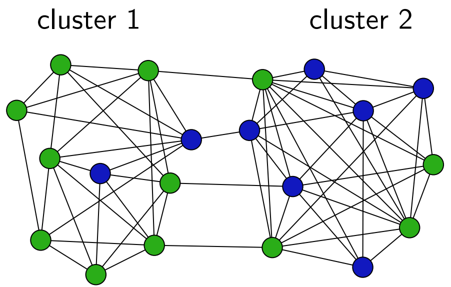

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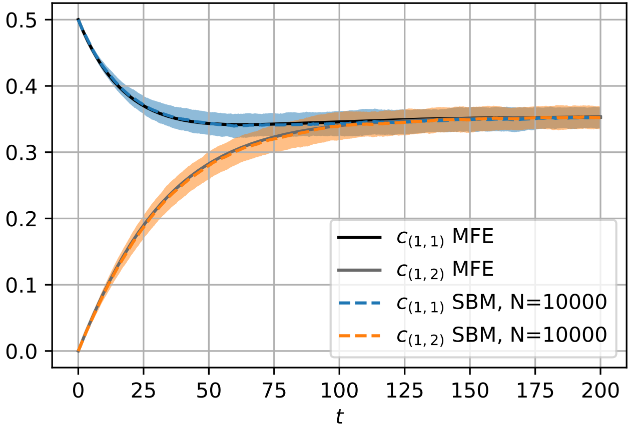


mean-field equation (L et al.) (Condition: $\forall k\ \exists k' : p_{k,k'} \gg \log(N)/N$)

$$\frac{d}{dt}c(t) = \sum_{(m,k)\to n} c_{(m,k)}(t)\left(r_{m,n}\frac{\sum_{k'\in[K]} c_{(n,k')}(t)\ p_{k,k'}}{\bar{p}_k} + \tilde{r}_{m,n}\right)\left(e_{(n,k)} - e_{(m,k)}\right)$$

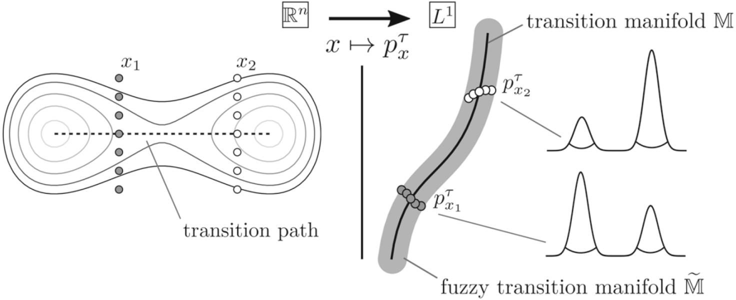
Outlook: Learning collective variables from data

- So far: $c \mathrel{\widehat{=}}$ concentration of states (in subgraphs).
- For other networks, a different choice for c might be required.

[1] A. Bittracher *et al.*, *Journal of Nonlinear Science* **31** (2020)

Outlook: Learning collective variables from data

- So far: $c \mathrel{\widehat{=}}$ concentration of states (in subgraphs).
- For other networks, a different choice for c might be required.
- Idea: Learn most important dynamical information (collective variables) from data (e.g., system trajectories) and choose c to extract this information
  - We use the *transition manifold approach*[1] for this purpose.



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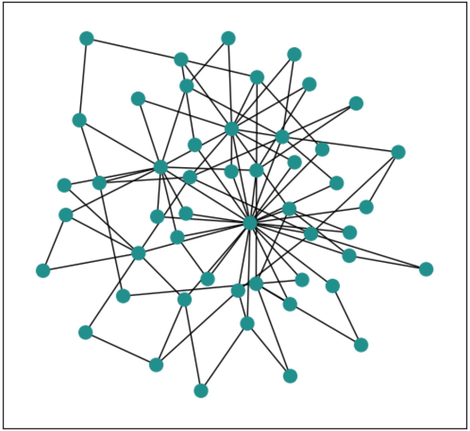
Example: Barabási-Albert network (scale-free)



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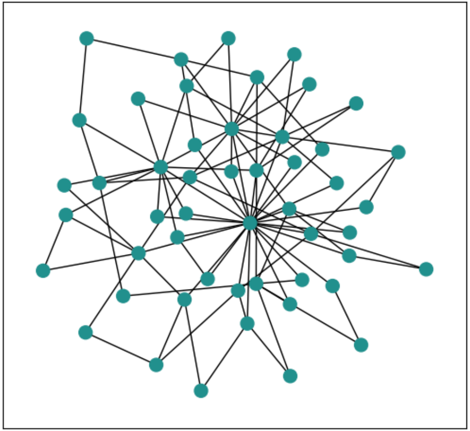
- Only $M = 2$ opinions, $x_i \in \{0, 1\}$.

$$c(x) := \frac{1}{\|\alpha\|_1} \sum_{i=1}^{N} \alpha_i x_i \quad \widehat{=} \text{ weighted concentration of states.}$$



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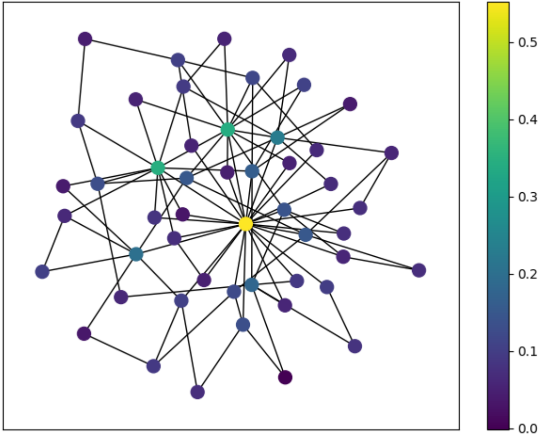

$\Rightarrow$ The weight α_i is given by the node degree d_i! Thus, c is the degree-weighted concentration.

References

- Marvin Lücke, Jobst Heitzig, Péter Koltai, Nora Molkenthin, Stefanie Winkelmann. Large population limits of Markov processes on random networks. arXiv:2210.02934, 2022.
- A. Bittracher, S. Klus, B. Hamzi, P. Koltai, and C. Schütte. Dimensionality reduction of complex metastable systems via kernel embeddings of transition manifolds. Journal of Nonlinear Science, 31, 2020.
- Thomas G. Kurtz. Strong approximation theorems for density dependent Markov chains. Stochastic Processes and their Applications, 1978.

Thank you for your attention!